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Amplification and disorder effects in a Kronig–Penney chain of active potentials

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Abstract. We report in this paper analytical and numerical results on the effect of amplification (due to non-Hermitian site potentials) on the transmission and reflection coefficients of a periodic one-dimensional Kronig–Penney lattice. A qualitative agreement is found with the tight-binding model where the transmission and reflection increase for small system lengths before strongly oscillating with a maximum at a certain length. For larger lengths the transmission decays exponentially at the same rate as in the growing region while the reflection saturates at a high value. However, the maximum transmission (and reflection) moves to larger system lengths and diverges in the limit of vanishing amplification instead of going to unity. In very large samples, it is anticipated that the presence of disorder and the associated length scale will limit this uninhibited growth in amplification. Also, there are other interesting competitive effects of the disorder and amplification giving rise to some non-monotonic behaviour in the peak of the transmission.

1. Introduction

Recently, there has been a lot of interest in non-Hermitian Hamiltonians and quantum phase transitions (typically localized to extended wavefunctions) in systems characterized by them. There are in general two classes of problems in this context: one in which the non-Hermiticity is in the non-local part [1, 2] and the other in which it is in the local part [3–8]. In the first category, one considers an imaginary vector potential added to the momentum operator in the Schrödinger Hamiltonian and this was shown to represent the physics of vortex lines pinned by columnar defects where the depinning is achieved [1] by a sufficiently high transverse magnetic field. In the case of a tight-binding Hamiltonian, the non-Hermiticity is introduced by a directed hopping in one of the directions (or more), and again in this case, it is intuitively clear that delocalization may be obtained in the preferred direction in the presence of randomness in the local potential even in 1D. In the second category (non-Hermiticity in the local term), an imaginary term is introduced in the one-body potential. It is well known from textbooks on quantum mechanics that depending on the sign of the imaginary term, this means the presence of a sink (absorber) or a source (amplifier) in the system. It may be noted that this second category does also have a counterpart in classical systems characterized by a Helmholtz (scalar) wave equation as well, where the practical application is in the studies of the effects of classical wave (light) localization due to back-scattering in the presence of an amplifying (lasing) medium that has a complex dielectric constant with spatial disorder in its real part [3, 6]. There is a common thread binding both problems though, namely that the spectrum for both becomes complex (the Hamiltonian being non-Hermitian or real non-symmetric), but can admit real eigenvalues as well. The common property is that the real eigenvalues represent localized states and the eigenvalues off the real lines represent extended states. That it is so in the first category has been shown in recent work, starting with that of Hatano and Nelson and followed by others' [1, 2]. For the second category with sources at each scatterer and in the absence of impurities, it seems counter-intuitive that there are localized solutions; but it has been shown in a simple way [8] that real eigenvalues are always localized. At present, there is no unified analysis of non-Hermiticity of both types. In the rest of this paper we will be concerned with non-Hermitian Hamiltonians of the second category only.

The interest in amplification effects of classical and quantum waves in disordered media has been strongly motivated by the recent experimental results on the amplification of light [9]. The amplification was shown to strongly enhance the coherent back-scattering and consequently increases the reflection [1–3]. These results on the reflection naturally lead one to expect an enhancement of the transmission in such amplifying systems. However, recently Sen [8] found for amplifying (non-Hermitian) periodic systems that the transmission coefficient starts increasing exponentially up to a certain length scale where it reaches its maximum, and then it oscillates strongly before decaying at larger length scales. The reflection seems to saturate to a constant value larger than unity. In this paper, we study in detail both analytically and numerically this scaling behaviour of the reflection and transmission within the framework of the Kronig–Penney model which differs from the tight-binding one in the fact that it is a continuous multiband model where the bandwidth depends on the potential strength while the tight-binding (TB) framework is a discrete single-band model where the bandwidth does not depend on the site energy. We compare our results with those obtained by Sen [8] within the tight-binding model and study the evolution of this behaviour with amplification. The effect of the competition between the amplification and disorder is also examined.

2. The model

We consider a non-interacting electron moving in a periodic system of δ-peak potentials having complex strength $\lambda = \lambda_0 + \mathrm{i}\eta$ where both λ_0 and η are constant numbers. By using the Poincaré map, the Schrödinger equation of this system can be transformed to the following discrete second-order equation [10, 11]:

$$\psi_{n+1} + \psi_{n-1} = \Omega\psi_n \tag{1}$$

where ψ_n stands for the electron wavefunction at the site n and

$$\Omega = 2\cos(\sqrt{E}) + \lambda\frac{\sin(\sqrt{E})}{\sqrt{E}}. \tag{2}$$

That is,

$$\Omega = 2\cos(k) \tag{3}$$

where k is the corresponding wavenumber. In the passive lattice (λ is real) the corresponding wavenumber is real in the allowed band ($|\Omega| \leqslant 2$) and the wavefunction is Bloch like, while in the band gap it is imaginary and the wavefunction becomes evanescent. In the case of an active lattice (λ is complex) the wavenumber becomes complex ($k = k_s + \mathrm{i}\gamma$) and equation (3) yields

$$2\cos(\sqrt{E}) + \lambda_0 \frac{\sin(\sqrt{E})}{\sqrt{E}} = (\mathrm{e}^{\gamma} + \mathrm{e}^{-\gamma})\cos(k_s) \tag{4}$$

$$\eta \frac{\sin(\sqrt{E})}{\sqrt{E}} = (\mathrm{e}^{-\gamma} - \mathrm{e}^{\gamma})\sin(k_s). \tag{5}$$

The main difference between the tight-binding model and this model is the direct dependence of the amplifying term γ on the electronic energy. If we restrict ourselves to the first band ($0 < k_s < \pi$) we see from (4) that γ is negative if η is positive. Obviously, in successive bands the sign of η must be changed alternately to get the same sign of γ. We note also that since we choose in our model, for the initial conditions of the discrete equation (1), an electron moving from the right-hand side to the left-hand side of the sample (see reference [11]) the amplification should occur for negative values of γ. Therefore the imaginary part of the potential should be positive in the first allowed band of the corresponding passive system. Indeed, the Hamiltonian is time-reversal invariant in the passive system but not in the active one, since the Hamiltonian is not Hermitian. From equation (1) the transmission coefficient can be obtained as

$$T = \frac{4\sin^2(\sqrt{E})\left|\mathrm{e}^{\mathrm{i}k_s}\mathrm{e}^{-\gamma} - \mathrm{e}^{\mathrm{i}k_s}\mathrm{e}^{\gamma}\right|^2}{\left|c\mathrm{e}^{\mathrm{i}k_s L}\mathrm{e}^{-\gamma L} - d\mathrm{e}^{-\mathrm{i}k_s L}\mathrm{e}^{\gamma L}\right|^2} \tag{6}$$

and the reflection coefficient as

$$R = \frac{\left|a\mathrm{e}^{\mathrm{i}k_s L}\mathrm{e}^{-\gamma L} - b\mathrm{e}^{-\mathrm{i}k_s L}\mathrm{e}^{\gamma L}\right|^2}{\left|c\mathrm{e}^{\mathrm{i}k_s L}\mathrm{e}^{-\gamma L} - d\mathrm{e}^{-\mathrm{i}k_s L}\mathrm{e}^{\gamma L}\right|^2} \tag{7}$$

where

$$a = \left[\mathrm{e}^{\mathrm{i}(k_s-\sqrt{E})}\mathrm{e}^{-\gamma} - 1\right]\left[\mathrm{e}^{\mathrm{i}(k_s+\sqrt{E})}\mathrm{e}^{-\gamma} - 1\right] \tag{8}$$

$$b = \left[\mathrm{e}^{-\mathrm{i}(k_s+\sqrt{E})}\mathrm{e}^{\gamma} - 1\right]\left[\mathrm{e}^{-\mathrm{i}(k_s-\sqrt{E})}\mathrm{e}^{\gamma} - 1\right] \tag{9}$$

$$c = 2 - \left[\mathrm{e}^{\mathrm{i}(k_s-\sqrt{E})}\mathrm{e}^{-\gamma} + \mathrm{e}^{-\mathrm{i}(k_s-\sqrt{E})}\mathrm{e}^{\gamma}\right] \tag{10}$$

$$d = 2 - \left[\mathrm{e}^{\mathrm{i}(k_s+\sqrt{E})}\mathrm{e}^{-\gamma} + \mathrm{e}^{-\mathrm{i}(k_s+\sqrt{E})}\mathrm{e}^{\gamma}\right]. \tag{11}$$

Since we are interested in studying the growth and decay regions of the transmission coefficient (and also the reflection coefficient) it turns out to be more efficient to write the coefficients c and d as follows:

$$c = \mathrm{e}^{-\mathrm{i}k_s\theta_c}\mathrm{e}^{\gamma L_0} \qquad d = \mathrm{e}^{\mathrm{i}k_s\theta_d}\mathrm{e}^{-\gamma L_1} \tag{12}$$

where

$$L_0 = \frac{\ln 2[\cosh(\gamma) - \cos(k_s - \sqrt{E})]}{\gamma} \qquad L_1 = -\frac{\ln 2[\cosh(\gamma) - \cos(k_s + \sqrt{E})]}{\gamma} \tag{13}$$

and $\theta_{c,d}$ are real phase parameters, which are expected to contribute to the oscillations of T, and behave linearly in γ for vanishing amplification. The transmission then reads

$$T = \frac{4\sin^2(\sqrt{E})\left|\mathrm{e}^{\mathrm{i}k_s}\mathrm{e}^{-\gamma} - \mathrm{e}^{-\mathrm{i}k_s}\mathrm{e}^{\gamma}\right|^2}{\left|\mathrm{e}^{\mathrm{i}(k_s L-\theta_c)}\mathrm{e}^{-\gamma(L-L_0)} - \mathrm{e}^{-\mathrm{i}(k_s L-\theta_d)}\mathrm{e}^{\gamma(L-L_1)}\right|^2}. \tag{14}$$

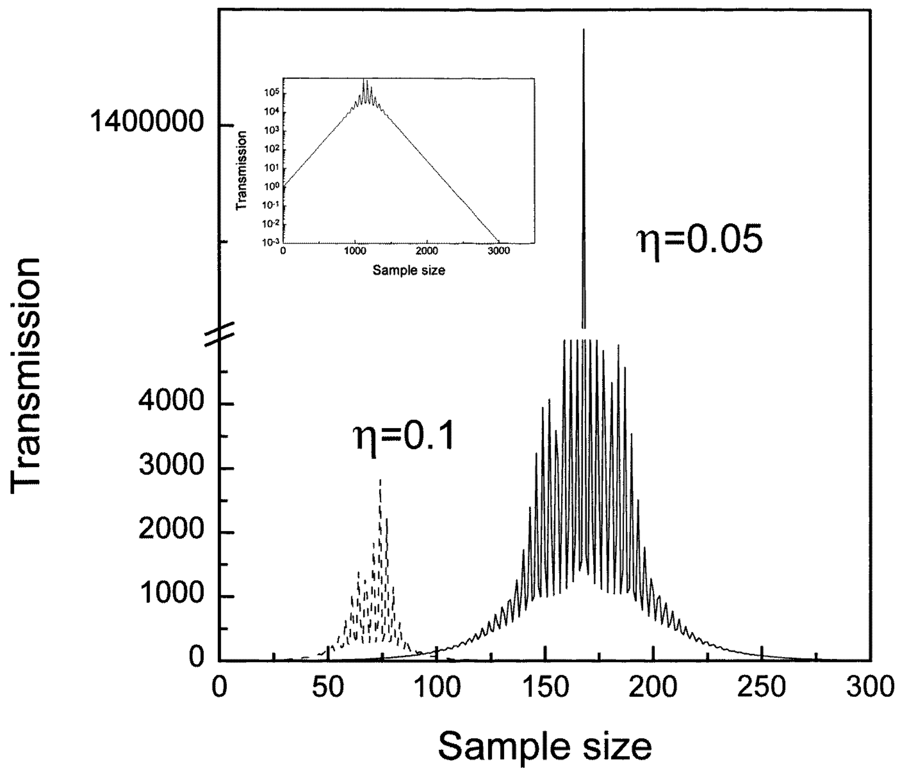

Figure 1. The transmission coefficient versus the sample size L for $\eta = 0.05$ (solid curve) and 0.1 (dashed curve). The inset is a semi-log plot produced to show the exponential growth for small sizes and the exponential asymptotic decay of the transmission.

3. Results and discussion

From equations (4) and (5) the initial amplification rate and the asymptotic (large-length-scale) decay rate γ depend explicitly on the complex potential strength and the energy. The analytical results hold both for an amplifying and an absorbing potential, and it is clearly seen that the asymptotic decay rate depends only on the magnitude of η and not on its sign. Thus the duality between the amplifying and the absorbing cases holds just as in the case of the tight-binding model [8]. The main influence on the asymptotic decay rate comes from the imaginary part η of the potential since a purely real potential does not induce amplification (or absorption), but all of the three parameters E, λ_0 and η contribute to this rate. In fact the variation of this rate is slower near the band edges ($\sqrt{E} = n\pi$) than at the band centre. In particular, at the band edges the system will be insensitive to the complex potential. Therefore, the energy and the real part of the potential fix the position of the wave vector inside the band. However, since we are interested in the effect of γ on the transmission and reflection, we can, without loss of generality, fix the energy and the real part of the potential. The amplification will then depend only on the imaginary part of the potential. This is also consistent with the case of the tight-binding model [8]. In the rest of the text we take $E = 1$ and $\lambda_0 = 0$. Furthermore, for the disordered case, we found with a tight-binding model [12] that the specific results may be quantitatively different for different values of energy and the complex potential, but the qualitative behaviour of the system does not change. In the present case also, our preliminary investigations indicate that to be the case. So, for the disordered case, we hold the energy to be $E = 1$ as before and let λ_0 be uniformly distributed in the domain $[-W/2, W/2]$ where W is considered to be the disorder strength. The decay of T for an absorbing chain is found from the above equations to be qualitatively similar to that for a disordered chain (with $\eta = 0$). Thus, nothing particularly interesting takes place for absorbers. But, as we discuss below, in the amplifying chain there is an interesting competition between amplification and disorder in the small-length-scale regime. For this reason our study below focuses on the amplification case where η must be positive. For numerical calculations, it is easier to use η instead of γ. In the limit of small γ we have a simple relation $\eta = -2\gamma$.

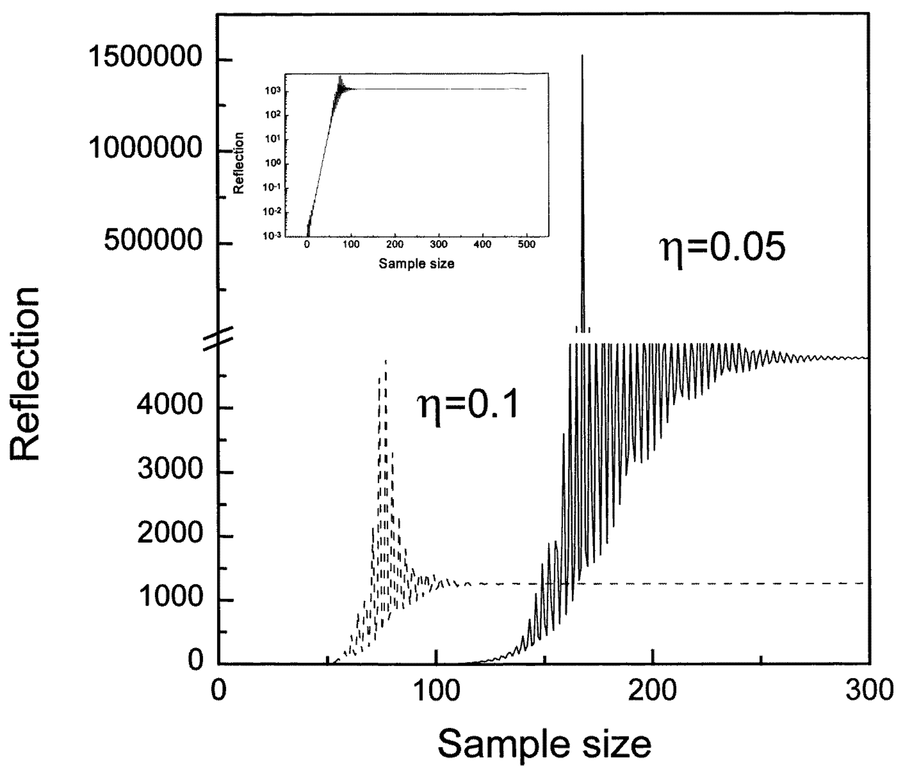

Figure 2. The reflection coefficient versus the sample size L for the same parameters as in figure 1. The inset is a semi-log plot produced to show the exponential growth of the reflection for small sizes.

In figure 1, we show the transmission as a function of the sample length for two different values of the amplification. It is shown that the transmission grows exponentially up to a certain length scale where oscillations set in and the transmission reaches its maximum value. For much larger lengths the transmission decays exponentially as in the case of an absorbing chain. A similar behaviour is shown in figure 2 for the reflection coefficient where, in contrast to the transmission case, for large lengths the back-scattering saturates (instead of decaying) to a high value of the reflection coefficient. This behaviour is in close agreement with that of the TB model [8] with a slight difference in the oscillatory region due to the different dependences of γ on η and the incident energy. This means that this effect is globally model independent. It is also shown in these figures that the maximum transmission and reflection increase on decreasing η and shift to higher sample lengths. Indeed, from equation (14) we see that when $L < L_1$ the coefficient d becomes dominant and then T behaves as $\exp(2|\gamma|L)$ while at asymptotically large lengths the coefficient c becomes dominant and the transmission decays as $\exp(-2|\gamma|L)$. In the oscillatory region the two coefficients c and d are of the same order of magnitude and the length at maximum transmission is approximately given by

$$L_{max} = \frac{1}{2\gamma} \ln \frac{\cosh(\gamma) - \cos(k_s - \sqrt{E})}{\cosh(\gamma) - \cos(k_s + \sqrt{E})}. \tag{15}$$

It is clear from this equation that L_{max} diverges for vanishing γ. However, since the maximum transmission must naturally be unity for a passive medium, T_{max} should not diverge for γ exactly equal to zero. Thus there is an infinite discontinuity at $\eta = 0$ which should turn towards a finite discontinuity at a finite disorder $W > 0$. In order to examine the limiting behaviour as $\eta \to 0$, let us use a perturbative treatment for $\eta \ll 1$. In this limit k_s tends to $\sqrt{E}$ as

$$k_s = \sqrt{E} + \frac{\gamma^2}{2\tan(1)} \tag{16}$$

and from equation (13) the lengths L_0 and L_1 are given by

$$L_0 = \frac{\ln(\gamma^2)}{\gamma} \qquad L_1 = -\frac{\ln(4\sin^2 k_s)}{\gamma}. \tag{17}$$

It may be noted that for very small η, T initially increases extremely slowly for small lengths and then it shoots up very quickly, when L becomes comparable to L_1, to a very large value of the transmission peak given by

$$T_{max} = \frac{1}{\gamma^2} \tag{18}$$

and the length at which this highest peak is obtained is given by

$$L_{max} = \frac{\ln(\gamma^2/\sin^2 k_s)}{2\gamma} \tag{19}$$

with the proviso that a negative value of L_{max} indicates that the peak occurs only at $L_{max} = 0$. Obviously this divergence with a discontinuity is a somewhat unexpected behaviour of the transmission. This is due to the fact that when $\eta \to 0^+$, L_{max} diverges more quickly than the amplification length scale $l_a = 1/\gamma$. Therefore γL_{max} will also diverge and, whenever γ is different from zero (positive), the current grows slowly up to a very large length scale and reaches very high values. One may note that the asymptotic reflection coefficient $R(L = \infty)$ also diverges as $\eta \to 0^+$ and has an infinite discontinuity at $\eta = 0$. Hence there is an extremely high amplification in the back-scattered wave for a very small η. For example, for a chain with $\eta = 10^{-4}$, $E = 1.0$, the transmission peak occurs at $L_{max} = 2.07 \times 10^5$, and $T_{max} = 2.87 \times 10^{10}$, and the asymptotic $R(L = \infty) = 1.13 \times 10^9$ which occurs at $L > L_{max}$. It is also seen from figures 1 and 2 that the period of the oscillations increases when γ decreases due to the increase of k_s. Before going on we would like to mention that all of the effects discussed above appear to be qualitatively similar to those in the TB model. For simplicity, if we take the Fermi energy at the band centre ($E = 0$), then we find that the maximum peak for transmission occurs at an $L_{max} \simeq (1/\eta)\ln(8\pi/\eta)$ which clearly diverges with $|\eta| \to 0$ and so does T_{max}.

However, the high amplitude of the largest peak in the transmission or the asymptotic value of the reflection coefficient even for very small amplification may not be observed experimentally since it occurs at very large sizes (see equation (19)) and the experimental realization of such perfect (disorder-free) systems is very difficult. Disorder, however small, would be present (in such a very-large-size system) and this may cut down strongly the divergences mentioned above. Now, as soon as one introduces disorder or, rather, takes care of the disorder, however small, the question regarding whether we should average or not comes up. On the one hand it is clear that experimentalists work on a typical sample, and not on a hypothetical 'average' sample. On the other hand, it may not be easy to keep a sample in the same state for a long time due to different types of relaxation process. Thus, the sample may change its characteristic with time if the characteristic under consideration is highly configuration dependent. Below we discuss both non-averaged and averaged transmission properties.

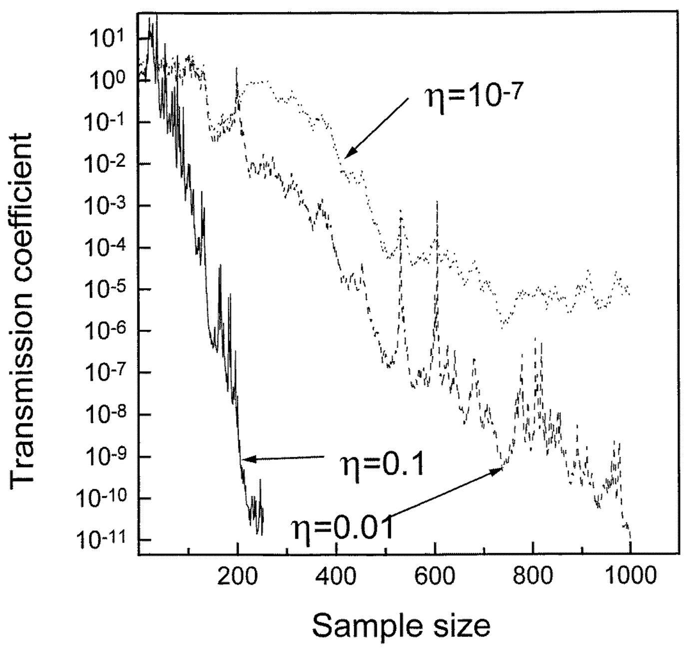

Figure 3. T versus L for a disordered lattice of λ_0 uniformly distributed between $-1/2$ and $1/2$ ($W = 1$) and $\eta = 0.1$ (solid curve), 0.01 (dashed curve) and 10^{-7} (dotted curve).

First, we discuss the transport properties for a particular configuration. For this part, we keep the disorder strength constant at $W = 1$. In figure 3 we show the effect of disorder on the transmission for different imaginary potentials. We see clearly that the disorder destroys the amplification at larger scales and shifts the maximum transmission to smaller lengths. The transmission fluctuations appearing in figure 3 increase with the amplification (η). As is well known, disorder introduces an exponential decay of the transmission with a rate $\gamma_{dis} = W^2/96E$ [13] where E is the energy of the incoming electron and γ_{dis} is the Lyapunov exponent due to the disorder. Stated differently, disorder introduces the localization length $\xi_{dis} = 1/\gamma_{dis}$ into the problem. For a small η, the length L_{max} up to which the exponential growth occurs in pure systems may be much larger than ξ_{dis}. So, in general, the transmission starts decaying due to disorder effects before it undergoes the

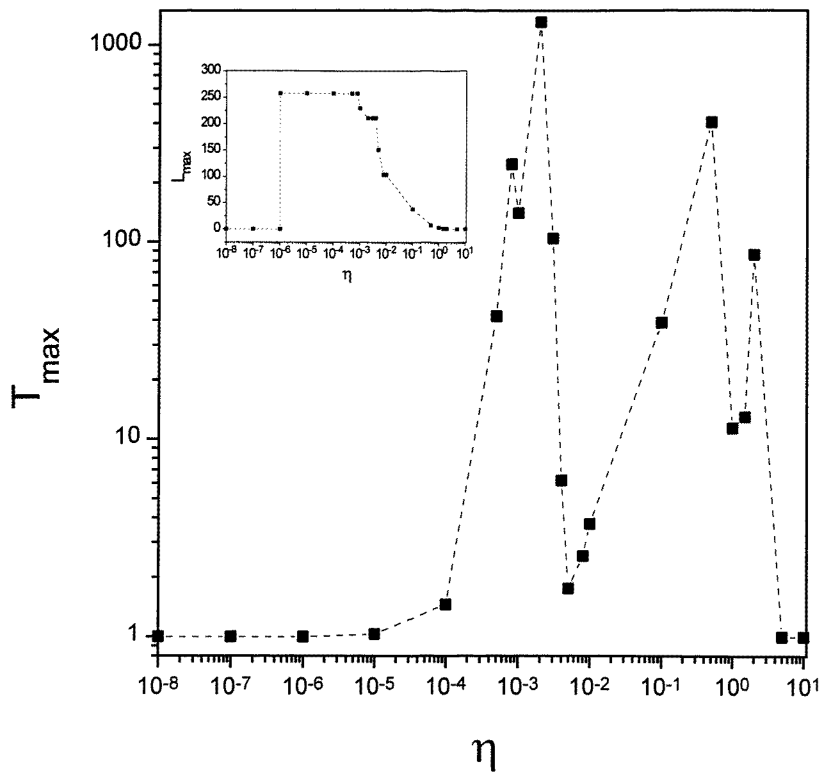

Figure 4. T_{max} versus η for the same configuration of the random real potential as in figure 3. The inset shows the corresponding length at maximum transmission (L_{max}) as a function of η. The dashed curve is only a guide for the eyes.

maximal amplification due to a non-zero η. Therefore, the divergence in T_{max} observed in periodic systems disappears with the disorder included as shown in figure 4. For very small η, T_{max} tends to the trivial constant value of unity with $L_{max} = 0$. But we have to remember that for $\xi_{dis} < L < L_{max}$ (for pure systems), there is a finely balanced competition between the amplification-dependent growth and the disorder-dependent decay which affects the transmission sensitively in this regime. As given by the above formula, for $W = 1$, $\xi_{dis} \simeq 100$. Yet, there is indeed a non-monotonic behaviour at much larger lengths, corresponding to some compensation between disorder and amplification. For $\eta \simeq 10^{-3}$, the transmission in general decays for $L > \xi_{dis}$ but only to pick up again at a still larger L, and one observes a peak of T_{max} (for the particular disorder configuration in figures 3 and 4) at $L \simeq 260$. This transmission peak seems to correspond to one of the Azbel resonances that becomes sensitively amplified by a tuned value of $\eta \simeq 10^{-3}$. We have actually checked that this resonance peak T_{max} occurs at the same L_{max} but becomes weaker both on increasing and on decreasing η around 0.001 as shown in figure 4 and thus T_{max} has a peak close to this special value of 0.001 for this particular configuration. In particular, if we decrease $\eta \to 10^{-6}$, the peak remains at $L_{max} \simeq 260$ while $T_{max} \to 1$ continuously. For $\eta < 10^{-6}$, the (local) peak transmission at $L \simeq 260$ becomes less than unity and hence the global $T_{max} = 1$ (a trivial constant) and L_{max} jumps back to the trivial value of zero discontinuously (see the inset of figure 4). Furthermore, as expected, we

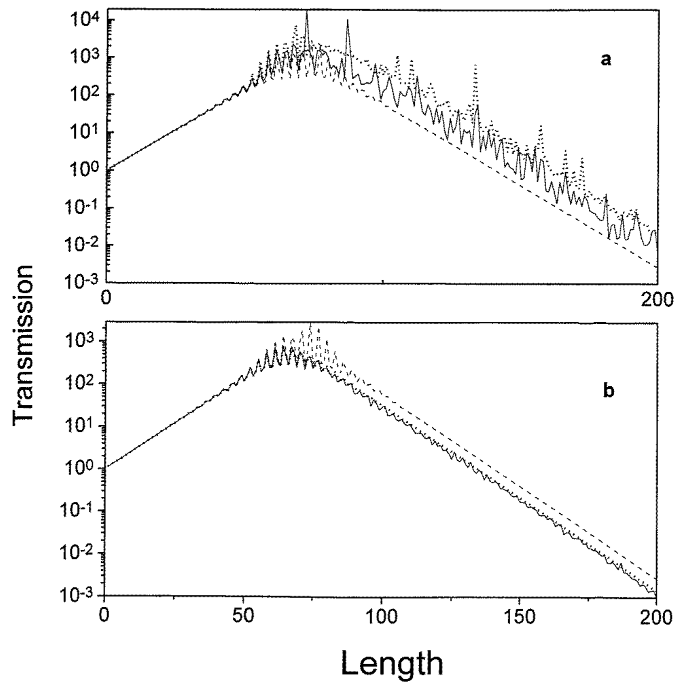

Figure 5. Transmission versus length for $E = 1$, $W = 0.01$ and $\eta = 0.1$ for an averaging over 100 samples (solid curve), over 10 000 samples (dotted curve) and without disorder (dashed curve); (a) averaging the quantity T itself and (b) averaging $\ln(T)$.

found that in other configurations, the peak in T_{max} at the special value of $\eta \simeq 10^{-3}$ as shown in figure 4 does not exist.

Next we discuss the above transport characteristics when sample averaging is performed. The question of what quantity to average becomes crucial now. In figure 5, we choose $E = 1$, $W = 0.01$ and $\eta = 0.1$ and show the transmission as a function of L (in a semi-log plot) by averaging in (a) the quantity T itself and in (b) the quantity $\ln T$. For comparison we have also shown the case without disorder by dashed lines. The full line is the result of averaging over 100 configurations and the dotted line is the same for 10 000 configurations in both cases. Whereas in figure 5(a) the average with 10 000 configurations lies higher than that with 100 configurations (both of them larger than for the pure case as well!), the logarithmic average shown in figure 5(b) is much more well behaved in every respect. The results shown here are consistent with the fact that all of the moments of the transmission and reflection diverge in the amplifying case [4, 5]. So, we restrict ourselves to logarithmic averaging and show in figure 6 the averaged T_{max} for two different η-values (0.01 with open squares and 0.1 with crosses). Due to the high sensitivity of T_{max} to η and for comparison purposes we have normalized the T_{max}-values to their non-disordered value ($W = 0$) in figure 6. Now, one expects that the non-monotonic behaviour as seen above should disappear since the Azbel resonances disappear after averaging. But, interestingly enough the fine tuning of the disorder and amplification is still at work, and some non-monotonic effects still survive. We have shown in the inset of figure 6 a magnified view of the y-axis around 1. Now we find that for the case of $\eta = 0.1$, there are some values of the disorder W around 0.01 at which the T_{max} is somewhat larger than its value for the

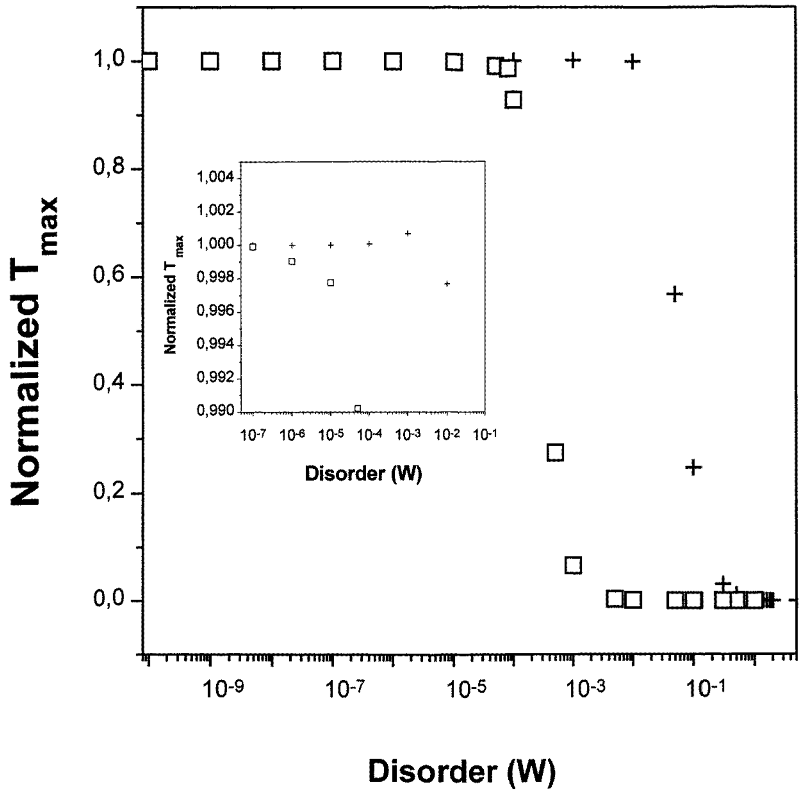

Figure 6. The normalized averaged maximum transmission T_{max} versus disorder for $\eta = 0.1$ (crosses) and $\eta = 0.01$ (open squares). The inset shows a magnified y-axis region between 0.99 and 1.01.

pure case. Furthermore, we could not find such an interesting non-monotonic behaviour for the case of $\eta = 0.01$ after a lot of searching, which means that even if it is there, it is probably very weak or lies in an extremely narrow region. At any rate, figure 6 shows clearly that the fine tuning of the disorder and amplification may lead to quite interesting and counter-intuitive results.

4. Conclusion

We have studied in this paper, within the framework of the Kronig–Penney model, the effect of amplification (due to non-Hermitian site potentials) on the transmission and reflection of a periodic as well as a disordered system. The analytical results hold for both the amplifying and the absorbing (pure) cases. Since the absorbing case shows nothing surprising but the amplifying case does, we have conducted numerical work only for the latter. The behaviour shown is in close agreement with that obtained in the tight-binding model [8]. Therefore, this effect seems to be generic, i.e., model independent. It may be noted, however, that the results for both the pure Kronig–Penney model and the tight-binding model imply that a diverging transmission peak (at a length $L_{max} \to \infty$) is obtained for a vanishing amplification ($\eta \to 0^+$) whereas in a passive system ($\eta = 0$) without disorder, the transmission coefficient is finite (unity). This infinite jump discontinuity of the T_{max}

or of L_{max} at $\eta = 0$ (for the pure system) is due to the divergence of L_{max} being faster than $1/\gamma$. This effect is probably experimentally unrealizable since large perfectly periodic samples cannot be grown and some disorder will always creep in to destroy this divergence. Indeed, we found that, in the presence of disorder, the T_{max}, as a non-monotonic function of η, decreases continuously to unity (and tends to saturate there for all $\eta < \eta_c$) but L_{max} decreases with a finite jump discontinuity to the trivial value of zero at the characteristic value η_c (dependent on the disorder and the specific configuration) of the imaginary part of the non-Hermitian potential. Since the transmission in general keeps growing due to amplification effects up to L_{max}, and in general keeps decaying due to disorder for lengths of the order of or larger than ξ_{dis}, there are interesting competition effects in the range $\xi_{dis} < L < L_{max}$. Qualitatively similar behaviours are observable at any energy or disorder [12], though the particular values of L_{max}, T_{max} etc may differ for different E- or W-values or specific configurations. More interestingly, the disorder and the amplification effects may work hand in hand (i.e., enhance each other) instead of competing with each other near those energies where the specific disordered configuration is near an Azbel resonance and $L < L_{max}$. In this case, both the disorder and the amplification effects tend to enhance the transmission vigorously. For example, for the case of figure 4, several peaks in T occur due to the competition effect, and the maximum peak at $\eta = 0.001$ seems to correspond to the enhancement effect of the disorder and amplification. However, the non-monotonic behaviours, e.g., the Azbel resonance, observed in particular realizations disappear on averaging over many realizations. Yet, some of the non-monotonic behaviour in figure 4 persists even after averaging in figure 6 (see the previous section). This is not well understood and hence should be extensively studied. Unlike in the tight-binding case [12], the bandwidth in the Kronig–Penney model depends on the scattering potential and this may give rise to some unexpected non-monotonic behaviour. Thus a generalization of this study to different electron energies and non-zero real parts of the potential (λ_0) could throw some important light, enhancing our understanding. Finally, for a further understanding of the surprising amplification effect in the periodic system, it would be interesting to study the amplification effect on the resonant tunnelling in a simple system of a double barrier which could give us a basis for understanding the periodic system.

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It was shown in this later article both analytically and numerically that the asymptotic decay constants are identical for an absorbing and an amplifying chain with the same magnitude for the strength of the non-Hermitian term. This somewhat surprising duality between the amplifying and the absorbing (ordered) cases was confirmed later in the following article:
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